数理解析研究所講究録 1255 巻 2002 年 1-21

On spectral and scattering theory for N-body Schrödinger operators in a constant magnetic field

神戸大学理学部　　足立　匡義 (Tadayoshi ADACHI)
Faculty of Science, Kobe University

1 Introduction

In this article, we study the spectral and scattering theory for N-body quantum systems in a constant magnetic field which contain some neutral particles.

The scattering theory for N-body quantum systems in a constant magnetic field has been studied by Gérard-Łaba [10, 11, 12] (see also [13]). But they have assumed that all particles in the systems are charged, that is, there is no neutral particle in the systems under consideration, even if the systems consist of only two particles (see also [17, 18]). Under this assumption, if there is no neutral proper subsystem, one has only to observe the behavior of all subsystems parallel to the magnetic field. Skibsted [24, 25] studied the scattering theory for N-body quantum systems in combined constant electric and magnetic fields, but his result needs the asymptotic completeness for the systems in a constant magnetic field.

Recently we studied the scattering theory for a two-body quantum system, which consists of one neutral and one charged particles, in a constant magnetic field (see [1]). Showing how to choose a conjugate operator for the Hamiltonian which governs the system was one of the ingredients in [1]. By virtue of this, we obtained the Mourre estimate and used it in order to obtain the so-called minimal velocity estimate which is one of useful propagation estimates.

Throughout this article, we consider an N-body quantum system which contains $N-1$ neutral particles and just one charged particle in a constant magnetic field. Our goal is to prove the asymptotic completeness of this system under short-range assumptions on the pair potentials. For achieving it, it is useful to obtain the Mourre estimate for the Hamiltonian which governs this system. The Mourre estimate is powerful also in studying spectral properties of the Hamiltonian. Finding a conjugate operator for the Hamiltonian is one of the ingredients in this article.

We consider a system of N particles moving in a given constant magnetic field $\boldsymbol{B} = (0,0,B) \in \boldsymbol{R}^3$, $B>0$. For $j=1,\ldots,N$, let $m_j>0$, $q_j \in \boldsymbol{R}$ and $x_j \in \boldsymbol{R}^3$ be the mass, charge and position vector of the j-th particle, respectively. Throughout this article, we assume that the last particle is charged and the rest are neutral, that is,

$$q_j = 0 \quad \text{if } 1 \leq j \leq N-1, \quad q_N \neq 0. \tag{1.1}$$

In particular, the total charge $q = \sum_j q_j$ of the system is non-zero in this case.

The total Hamiltonian for the system is defined by

$$\tilde{H} = \left(\sum_{j=1}^{N-1} \frac{1}{2m_j} D_{x_j}{}^2 \right) + \frac{1}{2m_N} (D_{x_N} - q_N \boldsymbol{A}(x_N))^2 + V \tag{1.2}$$

acting on $L^2(\boldsymbol{R}^{3\times N})$, where the potential V is the sum of the pair potentials $V_{jk}(x_j - x_k)$, that is,

$$V = \sum_{1\le j<k\le N} V_{jk}(x_j - x_k),$$

$D_{x_j} = -i\nabla_{x_j}$, $j = 1, \dots, N$, is the momentum operator of the j-th particle, and $\boldsymbol{A}(r)$ is the vector potential. Using the Coulomb gauge, the vector potential $\boldsymbol{A}(r)$ is given by

$$\boldsymbol{A}(r) = \frac{B}{2}(-r_2, r_1, 0), \quad r = (r_1, r_2, r_3). \tag{1.3}$$

As is well-known, it is easy to remove the center of mass motion of the system parallel to the field from the Hamiltonian $\tilde{H}$ (see e.g. [5]). In order to achieve it, we write the position x_j of the j-th particle for $x_j = (y_j, z_j)$ with $y_j \in \boldsymbol{R}^2$ and $z_j \in \boldsymbol{R}$. Moreover we identify the vector potential $\boldsymbol{A}(x_j) \in \boldsymbol{R}^3$ with $\boldsymbol{A}(y_j) \equiv (B/2)(-y_{j,2}, y_{j,1}) \in \boldsymbol{R}^2$ because $\boldsymbol{A}(x_j)$ can be written as $(\boldsymbol{A}(y_j), 0)$. Thus we study the spectral and scattering theory for the following Hamiltonian :

$$H = \left(\sum_{j=1}^{N-1} \frac{1}{2m_j} D_{y_j}{}^2 \right) + \frac{1}{2m_N} (D_{y_N} - q_N \boldsymbol{A}(y_N))^2 - \frac{1}{2}\Delta_{z^{a_{\max}}} + V \tag{1.4}$$

acting on $L^2(\boldsymbol{R}^{2\times N} \times Z^{a_{\max}})$, where $Z^{a_{\max}}$ is defined by

$$Z^{a_{\max}} = \left\{ z = (z_1, \dots, z_N) \in \boldsymbol{R}^N \;\middle|\; \sum_{j=1}^{N} m_j z_j = 0 \right\}$$

which is equipped with the metric

$$\langle z, \tilde{z} \rangle = \sum_{j=1}^{N} m_j z_j \tilde{z}_j, \quad |z|_1 = \sqrt{\langle z, z \rangle}$$

for $z = (z_1, \dots, z_N) \in \boldsymbol{R}^N$ and $\tilde{z} = (\tilde{z}_1, \dots, \tilde{z}_N) \in \boldsymbol{R}^N$, and $\Delta_{z^{a_{\max}}}$ is the Laplace-Beltrami operator on $Z^{a_{\max}}$.

Moreover, introducing the total pseudomomentum k_{total} of the system perpendicular to the field $\boldsymbol{B}$ which is defined by

$$k_{\text{total}} = \left(\sum_{j=1}^{N-1} D_{y_j} \right) + (D_{y_N} + q_N \boldsymbol{A}(y_N)), \tag{1.5}$$

one can remove the dependence on k_{total} from the Hamiltonian H : It is well-known that k_{total} commutes with H, and that since the total charge $q = q_N$ of this system is non-zero, the two components of the total pseudomomentum k_{total} cannot commute with each other, but satisfy the Heisenberg commutation relation (see e.g. [5]). Now we introduce the unitary operator

$$U = e^{-iy_{\text{cm}}\cdot qA(y_{\text{cc}})} e^{iqBy_{\text{cm},1}y_{\text{cm},2}/2} e^{iD_{y_{\text{cm},1}}D_{y_{\text{cm},2}}/(qB)} \tag{1.6}$$

on $L^2(\boldsymbol{R}^{2\times N} \times Z^{a_{\max}})$ with

$$y_{\text{cm}} = \frac{1}{M}\sum_{j=1}^{N} m_j y_j, \quad y_{\text{cc}} = \frac{1}{q}\sum_{j=1}^{N} q_j y_j, \tag{1.7}$$

where $M = \sum_j m_j$ is the total mass of the system. We note that $y_{\text{cc}} = y_N$ holds in this case. Then we obtain

$$U^* k_{\text{total},1} U = D_{y_{\text{cm},1}}, \quad U^* k_{\text{total},2} U = qBy_{\text{cm},1}, \tag{1.8}$$

and see that U^*HU is independent of $(D_{y_{\text{cm},1}}, qBy_{\text{cm},1})$ (see [10, 11, 12], [24, 25] and [1, 2]). Here the dot $\cdot$ means the usual Euclidean metric, and we wrote $k_{\text{total}} = (k_{\text{total},1}, k_{\text{total},2})$, $y_{\text{cm}} = (y_{\text{cm},1}, y_{\text{cm},2})$ and $D_{y_{\text{cm}}} = (D_{y_{\text{cm},1}}, D_{y_{\text{cm},2}})$. Thus one can identify the Hamiltonian U^*HU acting on $U^*L^2(\boldsymbol{R}^{2\times N} \times Z^{a_{\max}})$ with an operator acting on $\mathcal{H} = L^2(Y^{a_{\max}} \times \boldsymbol{R}_{y_{\text{cm},2}} \times Z^{a_{\max}})$, where $Y^{a_{\max}}$ is defined by

$$Y^{a_{\max}} = \left\{ y = (y_1, \dots, y_N) \in \boldsymbol{R}^{2\times N} \;\middle|\; \sum_{j=1}^{N} m_j y_j = 0 \right\}$$

which is equipped with the metric

$$\langle y, \tilde{y}\rangle = \sum_{j=1}^{N} m_j y_j \cdot \tilde{y}_j, \quad |y|_1 = \sqrt{\langle y, y\rangle}$$

for $y = (y_1, \dots, y_N) \in \boldsymbol{R}^{2\times N}$ and $\tilde{y} = (\tilde{y}_1, \dots, \tilde{y}_N) \in \boldsymbol{R}^{2\times N}$. We denote this reduced Hamiltonian acting on $\mathcal{H}$ by $\hat{H}$. It is a part of our goal to study the spectral theory for $\hat{H}$.

Now we state the assumptions on the pair potentials V_{jk}. For $r = (r_1, r_2, r_3) \in \boldsymbol{R}^3$, we denote (r_1, r_2) by $r_\perp$ and write $\nabla_{r_\perp} = \nabla_\perp$. For any interval $I \subset \boldsymbol{R}$, we denote the characteristic function of I on $\boldsymbol{R}$ by $\mathbf{1}_I$.

$(V.1)$ $V_{jk} = V_{jk}(r) \in L^2(\boldsymbol{R}^3) + L^\infty_\epsilon(\boldsymbol{R}^3)$ $(1 \le j < k \le N)$ is a real-valued function.
$(V.2)$ If j and k satisfy that $1 \le j < k \le N-1$, $r \cdot \nabla V_{jk}$ is $-\Delta$-bounded and satisfies

$$\left\| \mathbf{1}_{[1,\infty)}\left(\frac{|r|}{R}\right) r \cdot \nabla V_{jk}(-\Delta + 1)^{-1} \right\| = O(R^{-\mu}), \quad R \to \infty,$$

for some $\mu > 0$. Otherwise, that is, if l satisfies that $1 \le l \le N-1$, $\nabla_\perp V_{lN}$, $|\nabla_\perp V_{lN}|^2$ and $r \cdot \nabla V_{lN}$ are all $-\Delta$-bounded, and satisfy

$$\left\| \mathbf{1}_{[1,\infty)}\left(\frac{|r|}{R}\right) \nabla_\perp V_{lN}(-\Delta+1)^{-1} \right\| = O(R^{-\mu}), \quad R \to \infty,$$
$$\left\| \mathbf{1}_{[1,\infty)}\left(\frac{|r|}{R}\right) |\nabla_\perp V_{lN}|^2(-\Delta+1)^{-1} \right\| = O(R^{-\mu}), \quad R \to \infty,$$
$$\left\| \mathbf{1}_{[1,\infty)}\left(\frac{|r|}{R}\right) r \cdot \nabla V_{lN}(-\Delta+1)^{-1} \right\| = O(R^{-\mu}), \quad R \to \infty,$$

for some $\mu > 0$.
$(V.3)$ If j and k satisfy that $1 \le j < k \le N-1$, $(r \cdot \nabla)^2 V_{jk}$ is $-\Delta$-bounded. Otherwise, that is, if l satisfies that $1 \le l \le N-1$, $(\nabla_\perp)^2 V_{lN}$, $(r \cdot \nabla)^2 V_{lN}$, $\nabla_\perp (r \cdot \nabla V_{lN})$ and $r_\perp \cdot \nabla_\perp V_{lN}$ are all $-\Delta$-bounded.
(SR) V_{jk} satisfies that ∇V_{jk} is $-\Delta$-bounded and

$$\left\| \mathbf{1}_{[1,\infty)}\left(\frac{|r|}{R}\right) V_{jk}(-\Delta+1)^{-1} \right\| = O(R^{-\mu_{S1}}),$$
$$\left\| \mathbf{1}_{[1,\infty)}\left(\frac{|r|}{R}\right) \nabla V_{jk}(-\Delta+1)^{-1} \right\| = O(R^{-1-\mu_{S2}})$$

as $R \to \infty$, with $\mu_{S1} > 1$ and $\mu_{S2} > 0$.

Under these assumptions, the Hamiltonians H and $\hat{H}$ are self-adjoint.

To formulate the main result in this article precisely, we introduce some notations in many body scattering theory : A non-empty subset of the set $\{1, \dots, N\}$ is called a cluster. Let C_j, $1 \le j \le j_0$, be clusters. If $\cup_{1 \le j \le j_0} C_j = \{1, \dots, N\}$ and $C_j \cap C_k = \emptyset$ for $1 \le j < k \le j_0$, $a = \{C_1, \dots, C_{j_0}\}$ is called a cluster decomposition. We denote by $\#(a)$ the number of clusters in a. Let $\mathcal{A}$ be the set of all cluster decompositions. Suppose a, $b \in \mathcal{A}$. If b is a refinement of a, that is, if each cluster in b is a subset of a certain cluster in a, we say $b \subset a$, and its negation is denoted by $b \not\subset a$. Any cluster decomposition a can be regarded as a refinement of itself. If, in particular, b is a strict refinement of a, that is, if $b \subset a$ and $b \ne a$, we denote by $b \subsetneq a$. We identify the pair (j,k) with the $(N-1)$-cluster decomposition $\{\{j,k\}, \{1\}, \dots, \{\check{j}\}, \dots, \{\check{k}\}, \dots, \{N\}\}$. We denote by $a_{\max}$ and $a_{\min}$ the 1- and N-cluster decompositions, respectively. In this article, we often use the following notation

$$\mathcal{A}(a_{\max}) = \mathcal{A} \setminus \{a_{\max}\}.$$

We divide clusters into three types, that is, *neutral*, *charged* and *mixed* ones : Let $a = \{C_1, \dots, C_{\#(a)}\} \in \mathcal{A}$. Choose j_1 such that $1 \le j_1 \le \#(a)$ and $\{N\} \subset C_{j_1}$. Of course, this j_1 associated with a exists uniquely. If necessary, by renumbering the clusters in a, one can put $j_1 = \#(a)$ without loss of generality. C_j, $j = 1, \ldots, \#(a) - 1$, are called neutral clusters. If $C_{\#(a)} = \{N\}$, $C_{\#(a)}$ is called a charged cluster. Otherwise, $C_{\#(a)}$ is called a mixed cluster.

For $a \in \mathcal{A}$, the cluster Hamiltonian H_a is given by

$$\begin{aligned} H_a &= \left(\sum_{j=1}^{N-1} \frac{1}{2m_j} {D_{y_j}}^2 \right) + \frac{1}{2m_N}(D_{y_N} - q_j \boldsymbol{A}(y_N))^2 - \frac{1}{2}\Delta_{z^{a_{\max}}} + V^a, \\ V^a &= \sum_{(j,k)\subset a} V_{jk}(x_j - x_k) \end{aligned} \tag{1.9}$$

acting on $L^2(\boldsymbol{R}^{2\times N} \times Z^{a_{\max}})$. We define the innercluster Hamiltonian H^{C_j} on $L^2(\boldsymbol{R}^{2\times\#(C_j)} \times Z^{C_j})$ for each cluster $C_j = \{c_j(1), \ldots, c_j(\#(C_j))\}$ in a, where $\#(C_j)$ is the number of the elements in the cluster C_j : For a neutral cluster C_j, H^{C_j} is defined by

$$H^{C_j} = \left(\sum_{l\in C_j} \frac{1}{2m_l} {D_{y_l}}^2 \right) - \frac{1}{2}\Delta_{z^{C_j}} + V^{C_j}, \quad V^{C_j} = \sum_{\substack{\{l_1,l_2\}\subset C_j \\ l_1<l_2}} V_{l_1 l_2}(x_{l_1} - x_{l_2}). \tag{1.10}$$

For a charged cluster $C_{\#(a)}$, $H^{C_{\#(a)}}$ is defined by

$$H^{C_{\#(a)}} = \frac{1}{2m_N}(D_{y_N} - q_N \boldsymbol{A}(y_N))^2. \tag{1.11}$$

For a mixed cluster $C_{\#(a)}$, $H^{C_{\#(a)}}$ is defined by

$$\begin{aligned} H^{C_{\#(a)}} &= \left(\sum_{l\in C^{\mathrm{n}}_{\#(a)}} \frac{1}{2m_l} {D_{y_l}}^2 \right) + \frac{1}{2m_N}(D_{y_N} - q_N \boldsymbol{A}(y_N))^2 - \frac{1}{2}\Delta_{z^{C_{\#(a)}}} + V^{C_j}, \\ V^{C_j} &= \sum_{\substack{\{l_1,l_2\}\subset C_j \\ l_1<l_2}} V_{l_1 l_2}(x_{l_1} - x_{l_2}). \end{aligned} \tag{1.12}$$

where $C^{\mathrm{n}}_{\#(a)} = C_{\#(a)} \setminus \{N\}$. What we should emphasize here is that this $H^{C_{\#(a)}}$ is just the $\#(C_{\#(a)})$-body Hamiltonian under consideration. Here the configuration space Z^{C_j} is defined by

$$Z^{C_j} = \left\{ (z_{c_j(1)}, \ldots, z_{c_j(\#(C_j))}) \in \boldsymbol{R}^{\#(C_j)} \;\middle|\; \sum_{l=1}^{\#(C_j)} m_{c_j(l)} z_{c_j(l)} = 0 \right\},$$

which is equipped with the metric defined by

$$\langle \zeta, \tilde{\zeta} \rangle = \sum_{l=1}^{\#(C_j)} m_{c_j(l)} z_{c_j(l)} \tilde{z}_{c_j(l)}, \quad |\zeta|_1 = \sqrt{\langle \zeta, \zeta \rangle}$$

for $\zeta = (z_{c_j(1)}, \dots, z_{c_j(\#(C_j))}) \in \boldsymbol{R}^{\#(C_j)}$ and $\tilde{\zeta} = (\tilde{z}_{c_j(1)}, \dots, \tilde{z}_{c_j(\#(C_j))}) \in \boldsymbol{R}^{\#(C_j)}$, and $\Delta_{z^{C_j}}$ is the Laplace-Beltrami operator on Z^{C_j}. We also define two subspaces Z^a and Z_a of $Z^{a_{\max}}$ by

$$Z^a = \left\{ z \in Z^{a_{\max}} \;\middle|\; \sum_{l \in C_j} m_l z_l = 0 \text{ for each cluster } C_j \in a \right\}, \quad Z_a = Z^{a_{\max}} \ominus Z^a.$$

And we denote by Δ_{z^a} and Δ_{z_a} the Laplace-Beltrami operators on Z^a and Z_a, respectively. As is well-known, one can identify Z^a with $Z^{C_1} \oplus \cdots \oplus Z^{C_{\#(a)}}$. The cluster Hamiltonian H_a is decomposed into the sum of all the innercluster Hamiltonians H^{C_j} and $-\Delta_{z_a}/2$:

$$H_a = \left(\sum_{j=1}^{\#(a)} Id \otimes \cdots \otimes Id \otimes H^{C_j} \otimes Id \otimes \cdots \otimes Id \right) + Id \otimes \cdots \otimes Id \otimes \left(-\frac{1}{2} \Delta_{z_a} \right) \tag{1.13}$$

on $L^2(\boldsymbol{R}^{2\times N} \times Z^{a_{\max}}) = L^2(\boldsymbol{R}^{2\times\#(C_1)} \times Z^{C_1}) \otimes \cdots \otimes L^2(\boldsymbol{R}^{2\times\#(C_{\#(a)})} \times Z^{C_{\#(a)}}) \otimes L^2(Z_a)$.

We consider the sum of all the *neutral innercluster Hamiltonians* $H^{C_j}, j = 1 \dots, \#(a)-1$:

$$K(a) = \sum_{j=1}^{\#(a)-1} Id \otimes \cdots \otimes Id \otimes H^{C_j} \otimes Id \cdots \otimes Id \tag{1.14}$$

on $\mathcal{K}(a) = L^2(\boldsymbol{R}^{2\times\#(C_1)} \times Z^{C_1}) \otimes \cdots \otimes L^2(\boldsymbol{R}^{2\times\#(C_{\#(a)-1})} \times Z^{C_{\#(a)-1}})$. If one removes the center of mass motion perpendicular to the field $\boldsymbol{B}$ of this $(N - \#(C_{\#(a)}))$-body system from $K(a)$, the obtained Hamiltonian is an $(N - \#(C_{\#(a)}))$-body Schrödinger operator without external electromagnetic fields in the center of mass frame : We equip $\boldsymbol{R}^{2\times\#(C_j)}, j = 1, \dots, \#(a)-1$, with the metric

$$\langle \eta, \tilde{\eta} \rangle = \sum_{l=1}^{\#(C_j)} m_{c_j(l)} y_{c_j(l)} \cdot \tilde{y}_{c_j(l)}, \quad |\eta|_1 = \sqrt{\langle \eta, \eta \rangle}$$

for $\eta = (y_{c_j(1)}, \dots, y_{c_j(\#(C_j))}) \in \boldsymbol{R}^{2\times\#(C_j)}$ and $\tilde{\eta} = (\tilde{y}_{c_j(1)}, \dots, \tilde{y}_{c_j(\#(C_j))}) \in \boldsymbol{R}^{2\times\#(C_j)}$, and define two subspaces Y^{C_j} and Y_{C_j} of $\boldsymbol{R}^{2\times\#(C_j)}$ by

$$Y^{C_j} = \left\{ (y_{c_j(1)}, \dots, y_{c_j(\#(C_j))}) \in \boldsymbol{R}^{2\times\#(C_j)} \;\middle|\; \sum_{l=1}^{\#(C_j)} m_{c_j(l)} y_{c_j(l)} = 0 \right\},$$
$$Y_{C_j} = \boldsymbol{R}^{2\times\#(C_j)} \ominus Y^{C_j}.$$

And we put $X^{C_j} = Y^{C_j} \times Z^{C_j}$ and $X^{a,n} = X^{C_1} \times \cdots \times X^{C_{\#(a)-1}}$, and define two subspaces $Y^{a,n}$ and $Y_{a,n}$ of $\boldsymbol{R}^{2\times(N-\#(C_{\#(a)}))}$ by $Y^{a,n} = Y^{C_1} \times \cdots \times Y^{C_{\#(a)-1}}$ and $Y_{a,n} = \boldsymbol{R}^{2\times(N-\#(C_{\#(a)}))} \ominus Y^{a,n}$ which are equipped with the metric $\langle \, , \, \rangle$. Then $K(a)$ can be decomposed into

$$K(a) = K^a \otimes Id + Id \otimes \left(-\frac{1}{2} \Delta_{y_{a,n}} \right) \tag{1.15}$$

on $\mathcal{K}(a) = L^2(X^{a,\mathrm{n}}) \otimes L^2(Y_{a,\mathrm{n}})$, where $\Delta_{y_{a,\mathrm{n}}}$ is the Laplace-Beltrami operator on $Y_{a,\mathrm{n}}$. As we mentioned above, this Hamiltonian K^a is an $(N - \#(C_{\#(a)}))$-body Schrödinger operator without external electromagnetic fields in the center of mass frame. Thus we have

$$\begin{aligned} H_a =& K^a \otimes Id \otimes Id \otimes Id + Id \otimes H^{C_{\#(a)}} \otimes Id \otimes Id \\ &+ Id \otimes Id \otimes \left(-\frac{1}{2}\Delta_{y_{a,\mathrm{n}}}\right) \otimes Id + Id \otimes Id \otimes Id \otimes \left(-\frac{1}{2}\Delta_{z_a}\right) \end{aligned} \tag{1.16}$$

on $L^2(\boldsymbol{R}^{2\times N} \times Z^{a_{\max}}) = L^2(X^{a,\mathrm{n}}) \otimes L^2(\boldsymbol{R}^{2\times\#(C_{\#(a)})} \times Z^{C_{\#(a)}}) \otimes L^2(Y_{a,\mathrm{n}}) \otimes L^2(Z_a)$. Denoting by $\tilde{P}^a$ and $\hat{P}^a$ the eigenprojections for K^a on $L^2(X^{a,\mathrm{n}})$ and for $H^{C_{\#(a)}}$ on $L^2(\boldsymbol{R}^{2\times\#(C_{\#(a)})} \times Z^{C_{\#(a)}})$, respectively, we put

$$P^a = \tilde{P}^a \otimes \hat{P}^a \otimes Id \otimes Id$$

on $L^2(\boldsymbol{R}^{2\times N} \times Z^{a_{\max}}) = L^2(X^{a,\mathrm{n}}) \otimes L^2(\boldsymbol{R}^{2\times\#(C_{\#(a)})} \times Z^{C_{\#(a)}}) \otimes L^2(Y_{a,\mathrm{n}}) \otimes L^2(Z_a)$.

Then the usual wave operators $W_a^{\pm}$, $a \in \mathcal{A}(a_{\max})$, are defined by

$$W_a^{\pm} = \operatorname*{s-lim}_{t\to\pm\infty} e^{itH} e^{-itH_a} P^a. \tag{1.17}$$

The main result of this article is the following theorem.

Theorem 1.1. *Assume that* (V.1), (V.2), (V.3) *and* (SR) *are fulfilled. Then the usual wave operators* $W_a^{\pm}$, $a \in \mathcal{A}(a_{\max})$, *exist and are asymptotically complete*

$$L_c^2(H) = \sum_{a\in\mathcal{A}(a_{\max})} \oplus \, Ran\, W_a^{\pm}.$$

Here $L_c^2(H)$ *is the continuous spectral subspace of the Hamiltonian* H.

The problem of the asymptotic completeness for N-body quantum systems has been studied by many mathematicians and they have succeeded. For example, for N-body Schrödinger operators without external electromagnetic fields, this problem was first solved by Sigal-Soffer [22] for a large class of short-range potentials, and some alternative proofs appeared (see e.g. Graf [14] and Yafaev [26]). On the other hand, for the long-range case, Dereziński [7] solved this problem with arbitrary N for the class of potentials decaying like $O(|x_j - x_k|^{-\mu_L})$ with some $\mu_L > \sqrt{3} - 1$ (see also e.g. [8]). As for the results for the systems in external electromagnetic fields, see e.g. the references in [8] and [13].

Throughout this article, we assume that the number of charged particles L in the system under consideration is just one. In the case when $L \geq 2$, by virtue of the constant magnetic field, the physical situation in $\boldsymbol{R}^3$ seems quite different from the one in $\boldsymbol{R}^2$: Imagine N-body quantum scattering pictures both in $\boldsymbol{R}^3$ and in $\boldsymbol{R}^2$ under the influence of a constant magnetic field. Suppose that the last L particles are charged and cannot form any neutral clusters. Put $C^{\mathrm{n}} = \{1, \dots, N-L\}$ and $C^{\mathrm{c}} = \{N-L+1, \dots, N\}$, and introduce the set of cluster decompositions

$$\mathcal{B} = \left\{a = \{C_1, \dots, C_{\#(a)}\} \in \mathcal{A} \mid C^{\mathrm{c}} \subset C_{\#(a)}\right\}$$

with renumbering the clusters in a if necessary. For simplicity of the argument below, we suppose that the pair potentials are "short-range". As in the case when $L = 1$, one can also introduce the Hamiltonian H, cluster Hamiltonians H_a and the wave operators $W_a^{\pm}$. Then one expects that the statement of the asymptotic completeness says that

$$L_c^2(H) = \sum_{a \in \mathcal{A}(a_{\max})} \oplus \, Ran \, W_a^{\pm}$$

when the space dimension is three. As is well-known, it is equivalent to that the time evolution of any scattering state $\psi \in L_c^2(H)$ is asymptotically represented as

$$e^{-itH}\psi = \sum_{a \in \mathcal{A}(a_{\max})} e^{-itH_a} P^a \psi_a^{\pm} + o(1) \quad \text{as } t \to \pm\infty \tag{1.18}$$

with some $\psi_a^{\pm} \in L^2(\boldsymbol{R}^{2 \times N} \times Z^{a_{\max}})$. We note that each summand $e^{-itH_a} P^a \psi_a^{\pm}$ describes the motion of the particles in which those in the clusters in a form bound states and the centers of mass of the clusters in a move freely. Since the motion of the particles parallel to the magnetic field $\boldsymbol{B}$ is not influenced by $\boldsymbol{B}$, we need take a superposition of $e^{-itH_a} P^a \psi_a^{\pm}$ whose index a ranges in the whole of $\mathcal{A}(a_{\max})$ in general, as in the case when H is a usual N-body Schrödinger operators without external electromagnetic fields.

On the other hand, when the space dimension is two, the statement of the asymptotic completeness may be

$$L_c^2(H) = \sum_{a \in \mathcal{B}(a_{\max})} \oplus \, Ran \, W_a^{\pm},$$

where $\mathcal{B}(a_{\max}) = \mathcal{B} \setminus \{a_{\max}\} \subset \mathcal{A}(a_{\max})$. This says that the time evolution of any scattering state $\psi \in L_c^2(H)$ is asymptotically represented by a superposition of $e^{-itH_a} P^a \psi_a^{\pm}$, $a \in \mathcal{B}(a_{\max})$, which particularly describes the particles in the only charged cluster $C_{\#(a)}$ in $a \in \mathcal{B}(a_{\max})$ form bound states. The reason why we should take this $\mathcal{B}(a_{\max})$ instead of $\mathcal{A}(a_{\max})$ is as follows : All charged particles are bound in the directions perpendicular to the magnetic field $\boldsymbol{B}$ by the influence of $\boldsymbol{B}$, because they cannot form any neutral clusters. So one expects that the distance among all charged particles is bounded with respect to time t, and one can suppose that all charged particles belong to the same cluster. Hence we need not consider cluster decompositions $a \in \mathcal{A}(a_{\max})$ which have at least two charged clusters. Moreover, neutral particles can move freely without being influenced by the magnetic field $\boldsymbol{B}$ even when the space dimension is two. Thus one should study the motion of particles in the directions perpendicular to $\boldsymbol{B}$ more carefully in the case when $L \geq 2$. Recently we proved the existence of a conjugate operator for the reduced Hamiltonian $\hat{H}$ and the Mourre estimate also in this case, under the additional assumption that *the interactions between neutral and charged particles are finite-range* (see [3]). Though we assumed that the space dimension was three in [3], the proof is valid also in the case when the space dimension is two, by virtue of that they are finite-range.

Now what we would like to emphasize here is that the case in this article, that is, the case when $L = 1$ is the unique one in which

$$\mathcal{B}(a_{\max}) = \mathcal{A}(a_{\max})$$

holds, because $C^{c} = \{N\}$ only when $L = 1$. In fact, our argument can also be applied to studying the problem in $\boldsymbol{R}^2$ when $L = 1$, because the motion of the only charged particle in the directions perpendicular to $\boldsymbol{B}$ can be controlled by the total pseudomomentum k_{total} which does commute with the Hamiltonian H. This fact is a key in order to prove the main result.

2 The Mourre estimate

In this section, we find a conjugate operator for the Hamiltonian $\hat{H}$.

First we define the set of thresholds Θ for H (or $\hat{H}$) by induction in the number of neutral particles in the system. If $N = 2$, we put $\Theta = \tau_2$ (see [1]). Here

$$\tau_N = \left\{ \frac{|q_N|B}{m_N}\left(n + \frac{1}{2}\right) \,\middle|\, n \in \boldsymbol{N} \cup \{0\} \right\}. \tag{2.1}$$

Next let $N \geq 3$ and suppose that the sets of thresholds are defined for all k-body systems in which the number of charged particles is just one, with $2 \leq k \leq N-1$. Let $a = \{C_1, \dots, C_{\#(a)}\} \in \mathcal{A}(a_{\max})$ with $\{N\} \subset C_{\#(a)}$. As we emphasized above, if $C_{\#(a)}$ is mixed, $H^{C_{\#(a)}}$ is just the $\#(C_{\#(a)})$-body Hamiltonian under consideration. Then one can define the set of thresholds $\tau_{a,\mathrm{c}}$ for $H^{C_{\#(a)}}$ by the assumption of induction. Here it seems convenient that in the case when $C_{\#(a)}$ is charged, one puts $\tau_{a,\mathrm{c}} = \emptyset$. Put $\sigma_{a,\mathrm{c}} = \sigma_{\mathrm{pp}}(H^{C_{\#(a)}})$. Next we consider $K(a)$ on $\mathcal{K}(a)$. As we noted above, (1.15) holds, and K^a is an $(N - \#(C_{\#(a)}))$-body Schrödinger operator without external electromagnetic fields in the center of mass frame. Thus one can define the set of thresholds $\tau_{a,\mathrm{n}}$ for K^a as in the usual way. Put $\sigma_{a,\mathrm{n}} = \sigma_{\mathrm{pp}}(K^a)$. And set $\tilde{\tau}_{a,\mathrm{n}} = \tau_{a,\mathrm{n}} \cup \sigma_{a,\mathrm{n}}$ and $\tilde{\tau}_{a,\mathrm{c}} = \tau_{a,\mathrm{c}} \cup \sigma_{a,\mathrm{c}}$. Now we define the set of thresholds Θ for H (or $\hat{H}$) by

$$\Theta = \bigcup_{a \in \mathcal{A}(a_{\max})} (\tilde{\tau}_{a,\mathrm{n}} + \tilde{\tau}_{a,\mathrm{c}}). \tag{2.2}$$

Now we find the origin operator A of a conjugate operator $\hat{A}$ for the Hamiltonian $\hat{H}$. In order to achieve it, we recall the argument in [1] for a two-body system. Begin with the following self-adjoint operator A_1 on $L^2(\boldsymbol{R}^{2\times 2} \times Z^{a_{\max}})$ for H :

$$A_1 = \frac{1}{2}\{(\langle z^{a_{\max}}, D_{z^{a_{\max}}}\rangle + \langle D_{z^{a_{\max}}}, z^{a_{\max}}\rangle) + (y_1 \cdot D_{y_1} + D_{y_1} \cdot y_1)\}. \tag{2.3}$$

Putting $H_0 = H_{a_{\min}}$, one can obtain the following commutation relation by a straightforward computation :

$$i[H_0, A_1] = -\Delta_{z^{a_{\max}}} + \frac{1}{m_1} {D_{y_1}}^2 = 2\left(H_0 - \frac{1}{2m_2}(D_{y_2} - q_2 \boldsymbol{A}(y_2))^2\right). \tag{2.4}$$

As is well-known, the spectrum of the last term consists of the Landau levels τ_2. The commutation relation (2.4) seems nice for studying the spectral theory for the reduced Hamiltonian $\hat{H}$. However, since A_1 does not commute with k_{total}, $U^* A_1 U$ cannot be reduced on $\mathcal{H}$. In order to overcome this difficulty, we introduce the self-adjoint operator $\hat{A}_1$ on $\mathcal{H}$, which is obtained by removing the dependence on the total pseudomomentum $(D_{y_{\text{cm},1}}, qBy_{\text{cm},1})$ from the operator $U^* A_1 U$. This $\hat{A}_1$ is a conjugate operator for the reduced Hamiltonian $\hat{H}$. In [1], using the relative coordinates and the center of mass coordinates, we obtained this $\hat{A}_1$, but its representation was slightly complicated and unsuitable for generalizations to N-body systems. Now we review the argument in [1] : We see that the self-adjoint operator $U(\hat{A}_1 \otimes Id)U^*$ on $L^2(\boldsymbol{R}^{2\times 2} \times Z^{a_{\max}}) = U(\mathcal{H} \otimes L^2(\boldsymbol{R}_{y_{\text{cm},1}}))$ can be written as

$$U(\hat{A}_1 \otimes Id)U^* = \frac{1}{2}\{(\langle z^{a_{\max}}, D_{z^{a_{\max}}}\rangle + \langle D_{z^{a_{\max}}}, z^{a_{\max}}\rangle) + (w_1 \cdot D_{y_1} + D_{y_1} \cdot w_1)\} \tag{2.5}$$

with

$$w_1 = y_1 - \gamma_{\text{cc}}, \quad \gamma_{\text{cc}} = -\frac{2}{qB^2}\boldsymbol{A}(k_{\text{total}}), \tag{2.6}$$

where Id is the identity operator on $L^2(\boldsymbol{R}_{y_{\text{cm},1}})$, and γ_{cc} is called the center of orbit of the center of charge of the system (see [5] and [11, 12, 13]). In this case, one knows that $q = q_2$, of course. Now we note that

$$y_{\text{cc}} - \gamma_{\text{cc}} = y_2 - \gamma_{\text{cc}} = \frac{2}{qB^2}\boldsymbol{A}(D_{y_1} + (D_{y_2} - q_2\boldsymbol{A}(y_2))) \tag{2.7}$$

is H-bounded. Since $y_{\text{cc}} - \gamma_{\text{cc}}$ commutes with the total pseudomomentum k_{total}, $U^*(y_{\text{cc}} - \gamma_{\text{cc}})U$ is $\hat{H}$-bounded, where we regarded $U^*(y_{\text{cc}} - \gamma_{\text{cc}})U$ as the reduced one acting on $\mathcal{H}$. We notice that one can write

$$i[V_{12}, \hat{A}_1] = -(x_1 - x_2) \cdot \nabla V_{12}(x_1 - x_2) - (U^*(y_2 - \gamma_{\text{cc}})U) \cdot \nabla_\perp V_{12}(x_1 - x_2)$$

on $\mathcal{H}$, noting that V_{12} commutes with k_{total}. Under the assumptions $(V.1)$ and $(V.2)$, we see that $(\hat{H}_0+1)^{-1}i[V_{12},\hat{A}](\hat{H}_0+1)^{-1}$ is bounded on $\mathcal{H}$, and that for any $\varepsilon>0$ and real-valued $f\in C_0^\infty(\boldsymbol{R})$ there exists a compact operator K on $\mathcal{H}$ such that

$$f(\hat{H})i[V_{12},\hat{A}_1]f(\hat{H})\geq -\varepsilon f(\hat{H})^2+K$$

holds. Here we used the fact that $U^*(y_{\text{cc}}-\gamma_{\text{cc}})U$ is $\hat{H}$-bounded, which was mentioned above. Since both D_{y_1} and k_{total} commute with H_0, it is clear that

$$i[\hat{H}_0,\hat{A}_1]=2\left(\hat{H}_0-U^*\left\{\frac{1}{2m_2}(D_{y_2}-q_2\boldsymbol{A}(y_2))^2\right\}U\right) \tag{2.8}$$

holds by virtue of (2.4), where $\hat{H}_0$ and $U^*\{(1/2m_1)(D_{y_2}-q_2\boldsymbol{A}(y_2))^2\}U$ are the reduced operators acting on $\mathcal{H}$ of H_0 and $(1/2m_1)(D_{y_2}-q_2\boldsymbol{A}(y_2))^2$, respectively. By virtue of these two estimates, we obtained the desired Mourre estimate in [1] (see also Theorem 2.1 in this section).

Now we return to the present problem. We define the origin operator A of a conjugate operator $\hat{A}$ for the reduced Hamiltonian $\hat{H}$:

$$A=\frac{1}{2}\left\{(\langle z^{a_{\max}},D_{z^{a_{\max}}}\rangle+\langle D_{z^{a_{\max}}},z^{a_{\max}}\rangle)+\sum_{j=1}^{N-1}(w_j\cdot D_{y_j}+D_{y_j}\cdot w_j)\right\} \tag{2.9}$$

with

$$w_j=y_j-\gamma_{\text{cc}},\quad \gamma_{\text{cc}}=-\frac{2}{qB^2}\boldsymbol{A}(k_{\text{total}}),\quad j=1,\dots,N-1. \tag{2.10}$$

We see that A commutes with the total pseudomomentum k_{total}, by taking account of the fact that D_{y_j} and w_j, $j=1,\dots,N-1$, commute with k_{total}. Here we note that $q=q_N$ and $y_{\text{cc}}=y_N$ in this case, and that

$$y_{\text{cc}}-\gamma_{\text{cc}}=\frac{2}{qB^2}\boldsymbol{A}\left(\left(\sum_{j=1}^{N-1}D_{y_j}\right)+(D_{y_N}-q_N\boldsymbol{A}(y_N))\right) \tag{2.11}$$

is H_0-bounded. We also notice that $U^*(y_{\text{cc}}-\gamma_{\text{cc}})U$ is $\hat{H}_0$-bounded, since $y_{\text{cc}}-\gamma_{\text{cc}}$ commutes with k_{total} and is H_0-bounded as we mentioned just now, where we regarded $U^*(y_{\text{cc}}-\gamma_{\text{cc}})U$ as the reduced one acting on $\mathcal{H}$. Since D_{y_j}, $j=1,\dots,N-1$, and k_{total} all commute with H_0, it is clear that

$$i[H_0,A]=-\Delta_{z^{a_{\max}}}+\sum_{j=1}^{N-1}\frac{1}{m_j}D_{y_j}{}^2=2\left(H_0-\frac{1}{2m_N}(D_{y_N}-q_N\boldsymbol{A}(y_N))^2\right) \tag{2.12}$$

holds. And we define a conjugate operator $\hat{A}$ for the reduced Hamiltonian $\hat{H}$ as the reduced operator on $\mathcal{H}$ of A. The Nelson's commutator theorem guarantees the self-adjointness of A and $\hat{A}$ (see e.g. [21]). Moreover, by virtue of the fact that $U^*(y_{cc} - \gamma_{cc})U$ is $\hat{H}_0$-bounded, one can check that $(\hat{H}_0 + 1)^{-1}i[V, \hat{A}](\hat{H}_0 + 1)^{-1}$ is bounded on $\mathcal{H}$ in the same way as in the two-body case which we mentioned above, under the assumptions $(V.1)$ and $(V.2)$. We have only to keep in mind that $w_{j_1} - w_{j_2} = y_{j_1} - y_{j_2}$ with $1 \le j_1, j_2 \le N-1$.

Then we have the following main result of this section by virtue of the abstract Mourre theory (see e.g. [19] and [6]) and the HVZ theorem for the reduced Hamiltonian $\hat{H}$ (it is well-known that the HVZ theorem for H cannot hold, since H has the so-called Landau degeneracy which was proved in [5]) :

Theorem 2.1. *Suppose that the potential V satisfies the conditions $(V.1)$ and $(V.2)$. Put*

$$d(\lambda) = \mathrm{dist}(\lambda, \Theta \cap (-\infty, \lambda])$$

for $\lambda \ge \inf \Theta$, where Θ is as in (2.2). Then for any $\lambda \ge \inf \Theta$ and any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any real-valued $f \in C_0^\infty(\boldsymbol{R})$ supported in the open interval $(\lambda - \delta, \lambda + \delta)$, there exists a compact operator K on $\mathcal{H}$ such that

$$f(\hat{H})i[\hat{H}, \hat{A}]f(\hat{H}) \ge 2(d(\lambda) - \varepsilon)f(\hat{H})^2 + K \tag{2.13}$$

holds. Moreover, eigenvalues of $\hat{H}$ can accumulate only at Θ, and $\Theta \cup \sigma_{\mathrm{pp}}(\hat{H})$ is a closed countable set.

As for the proof, see [2].

In order to study the scattering theory for the Hamiltonian H, the following corollary seems useful, which follows from the fact that $\hat{H}$ is the reduced operator on $\mathcal{H}$ of H and a standard argument immediately (cf. [1]) :

Corollary 2.2. *Suppose that the potential V satisfies the conditions $(V.1)$ and $(V.2)$. Then for any $\lambda \in \boldsymbol{R} \setminus (\Theta \cup \sigma_{\mathrm{pp}}(H))$, there exist $\delta > 0$ and $c > 0$ such that for any real-valued $f \in C_0^\infty(\boldsymbol{R})$ supported in the open interval $(\lambda - \delta, \lambda + \delta)$,*

$$f(H)i[H, A]f(H) \ge cf(H)^2 \tag{2.14}$$

holds.

3 Propagation estimates

In this section, we introduce some propagation estimates which are useful for showing the asymptotic completeness for the system under consideration.

Throughout this section, we assume that the potential V satisfies the following condition (LR) as well as $(V.1)$, $(V.2)$ and $(V.3)$.

(LR) V_{jk} is decomposed as $V_{jk} = V_{jk,S} + V_{jk,L}$, where a real-valued $V_{jk,L} \in C^\infty(\boldsymbol{R}^3)$ such that $|\partial_r^\alpha V_{jk,L}(r)| \le C_\alpha \langle r\rangle^{-|\alpha|-\mu_L}$ with $0 < \mu_L \le 1$, and a real-valued $V_{jk,S}$ satisfies that $\nabla V_{jk,S}$ is $-\Delta$-bounded and

$$\left\| \mathbf{1}_{[1,\infty)}\left(\frac{|r|}{R}\right) V_{jk,S}(-\Delta+1)^{-1} \right\| = O(R^{-\mu_{S1}}),$$
$$\left\| \mathbf{1}_{[1,\infty)}\left(\frac{|r|}{R}\right) \nabla V_{jk,S}(-\Delta+1)^{-1} \right\| = O(R^{-1-\mu_{S2}})$$

as $R \to \infty$, with $\mu_{S1} > 1$ and $\mu_{S2} > 0$.

One can use this condition (LR) in the study of long-range scattering for N-body quantum systems in a constant magnetic field under the condition that the number of charged particles in the systems is only one. We note that by putting $V_L \equiv 0$, (LR) implies (SR).

Inspired by [1], we first introduce the configuration space $\mathcal{X} = \boldsymbol{R}^{2\times(N-1)} \times Z^{a_{\max}}$ which is equipped with the metric

$$\langle \Xi, \tilde{\Xi}\rangle = \left(\sum_{j=1}^{N-1} m_j y_j \cdot \tilde{y}_j\right) + \langle z^{a_{\max}}, \tilde{z}^{a_{\max}}\rangle, \quad |\Xi|_1 = \sqrt{\langle \Xi, \Xi\rangle}$$

for $\Xi = (y_1, \dots, y_{N-1}, z^{a_{\max}}) \in \mathcal{X}$ and $\tilde{\Xi} = (\tilde{y}_1, \dots, \tilde{y}_{N-1}, \tilde{z}^{a_{\max}}) \in \mathcal{X}$. We denote the velocity operator associated with Ξ by $p_\Xi = -i\nabla_\Xi$.

Now, for $a = \{C_1, \dots, C_{\#(a)}\} \in \mathcal{A}$ with $\{N\} \subset C_{\#(a)}$, we introduce two subspaces $\mathcal{X}^a$ and $\mathcal{X}_a$ of $\mathcal{X}$ as follows :

$$\mathcal{X}^a = \left\{ (y_1, \dots, y_{N-1}) \in \boldsymbol{R}^{2\times(N-1)} \;\middle|\; \sum_{k\in C_j} m_k y_k = 0 \text{ for any } j = 1, \dots, \#(a)-1 \right\} \times Z^a,$$
$$\mathcal{X}_a = \{ (y_1, \dots, y_{N-1}) \in \boldsymbol{R}^{2\times(N-1)} \mid y_{l_1} = y_{l_2} \text{ if } l_1,\, l_2 \in C_j, \text{ for any } j = 1, \dots, \#(a)-1\,; \; y_k = 0 \text{ if } k \in C_{\#(a)} \} \times Z_a.$$

We see that these two subspaces are mutually orthogonal, and that $\mathcal{X}^a \oplus \mathcal{X}_a = \mathcal{X}$. We denote by π^a and π_a the orthogonal projections of $\mathcal{X}$ onto $\mathcal{X}^a$ and $\mathcal{X}_a$, respectively. And we write $\Xi^a = \pi^a \Xi$ and $\Xi_a = \pi_a \Xi$. Denoting the velocity operators associated with Ξ^a and Ξ_a by $p_{\Xi^a} = -i\nabla_{\Xi^a}$ and $p_{\Xi_a} = -i\nabla_{\Xi_a}$, respectively, we see that $p_{\Xi^a} = \pi^a p_\Xi$ and $p_{\Xi_a} = \pi_a p_\Xi$. For $a,\, b \in \mathcal{A}$, we denote the smallest cluster decomposition $c \in \mathcal{A}$ with $a \subset c$ and $b \subset c$ by $a \cup b$, whose existence and uniqueness are well-known. Then we note that for $a,\, b \in \mathcal{A}$

$$\mathcal{X}_{a\cup b} = \mathcal{X}_a \cap \mathcal{X}_b$$

holds, which can be seen easily.

Now we can introduce the so-called Graf vector field as in [14] and [7] (see also [8]) :

Proposition 3.1. *There exist a smooth convex function $R(\Xi)$ on $\mathcal{X}$, bounded smooth functions $\tilde{q}_a(\Xi)$ and $q_a(\Xi)$, $a \in \mathcal{A}$, on $\mathcal{X}$ which satisfy the following : $\tilde{q}_a(\Xi)$ and $q_a(\Xi)$, $a \in \mathcal{A}$, have bounded derivatives. If $(j,k) \not\subset a$, $|\Xi^{(j,k)}|_1 \geq \sqrt{3r^{N-1}/10}$ holds on* $\operatorname{supp}\tilde{q}_a(\Xi)$ *and* $\operatorname{supp} q_a(\Xi)$. *In particular, if $(j,k) \not\subset a$ and $j < k < N$, there exists some $c > 0$ such that $|x_j - x_k| \geq c$ holds on* $\operatorname{supp}\tilde{q}_a(\Xi)$ *and* $\operatorname{supp} q_a(\Xi)$. *Moreover, one has*

$$\sum_{a\in\mathcal{A}} \tilde{q}_a(\Xi) \equiv 1, \quad \sum_{a\in\mathcal{A}} {q_a}^2(\Xi) \equiv 1,$$

$$\max\{{|\Xi|_1}^2, C_1\} \leq 2R(\Xi) \leq {|\Xi|_1}^2 + C_2 \quad \text{for some } C_1,\ C_2 > 0,$$

$$(\nabla_\Xi R)(\Xi) = \sum_{a\in\mathcal{A}} \Xi_a \tilde{q}_a(\Xi),$$

$$(\nabla_\Xi^2 R)(\Xi) \geq \sum_{a\in\mathcal{A}} \pi_a \tilde{q}_a(\Xi),$$

$$\langle \xi, (\nabla_\Xi^2 R)(\Xi)\xi\rangle - \langle \xi, (\nabla_\Xi R)(\Xi)\rangle - \langle (\nabla_\Xi R)(\Xi), \xi\rangle + 2R(\Xi) \geq \sum_{a\in\mathcal{A}} \tilde{q}_a(\Xi){|\xi_a - \Xi_a|_1}^2$$

for $\xi \in \mathcal{X}$, and that for any $a \in \mathcal{A}$, R depends on Ξ_a only in some neighborhood of $\mathcal{X}_a$. $\partial_\Xi^\alpha(2R(\Xi) - {|\Xi|_1}^2)$, $\partial_\Xi^\alpha(\langle \Xi, (\nabla_\Xi R)(\Xi)\rangle - {|\Xi|_1}^2)$ and $\partial_\Xi^\alpha(\langle \Xi, (\nabla_\Xi^2 R)(\Xi)\Xi\rangle - {|\Xi|_1}^2)$ are all bounded functions on $\mathcal{X}$, for any multi-index α.

Following the argument of [5], we introduce the creation operator β^* by using the total pseudomomentum $k_{\mathrm{total}} = (k_{\mathrm{total},1}, k_{\mathrm{total},2})$ as follows (see also [1]) :

$$\beta^* = \frac{1}{\sqrt{2}}\left(\frac{1}{qB}k_{\mathrm{total},2} - ik_{\mathrm{total},1}\right). \tag{3.1}$$

Here we took account of (1.8). In the argument below, we use the localization of the number operator $N_0 = \beta^*\beta$ in addition to the localization of the energy.

Now we show the following important propagation estimate, which was due to Graf [14] in the case of N-body Schrödinger operators without external electromagnetic fields (see also [7] and [8]).

Theorem 3.2. *Let $a \in \mathcal{A}$, $J \in C_0^\infty(\mathcal{X})$ be a cut-off function such that $J = 1$ on $\{\Xi \in \mathcal{X} \mid |\Xi|_1 \leq \theta\}$ and $J \geq 0$, and $f, h \in C_0^\infty(\boldsymbol{R})$ be real-valued. Suppose that* $\max\{(1+\mu_{S2})^{-1}, (1+\mu_L)^{-1}\} < \nu \leq 1$. *Then, for sufficiently large $\theta > 0$, there exists a constant $C > 0$ such that for any $\psi \in L^2(\boldsymbol{R}^{2\times N} \times Z^{a_{\max}})$*

$$\int_1^\infty \left\| \left|\frac{\Xi_a}{t} - p_{\Xi_a}\right|_1 q_a\left(\frac{\Xi}{t^\nu}\right) J\left(\frac{\Xi}{t}\right) f(H)h(N_0)e^{-itH}\psi\right\|^2 \frac{dt}{t} \leq C\|\psi\|^2$$

As for the proof, see [2].

When we take $\nu = 1$ in Theorem 3.2, one can obtain an improvement of Theorem 3.2 as follows :

Theorem 3.3. *Let $a \in \mathcal{A}$, $J \in C_0^\infty(\mathcal{X})$ be a cut-off function such that $J = 1$ on $\{\Xi \in \mathcal{X} \mid |\Xi|_1 \leq \theta\}$ and $J \geq 0$, and $f, h \in C_0^\infty(\boldsymbol{R})$ be real-valued. Then, for sufficiently large $\theta > 0$, there exists a constant $C > 0$ such that for any $\psi \in L^2(\boldsymbol{R}^{2\times N} \times Z^{a_{\max}})$*

$$\int_1^\infty \left\| \left| \frac{\Xi_a}{t} - p_{\Xi_a} \right|_1^{1/2} q_a\left(\frac{\Xi}{t}\right) J\left(\frac{\Xi}{t}\right) f(H)h(N_0)e^{-itH}\psi \right\|^2 \frac{dt}{t} \leq C\|\psi\|^2$$

holds.

As for the proof, see [2].

Next we introduce the following maximal velocity estimate.

Proposition 3.4. *For any real-valued $f \in C_0^\infty(\boldsymbol{R})$ there exists $M > 0$ such that for any $M_2 > M_1 \geq M$,*

$$\int_1^\infty \left\| \mathbf{1}_{[M_1,M_2]}\left(\frac{|\Xi|_1}{t}\right) f(H)e^{-itH}\psi \right\|^2 \frac{dt}{t} \leq C\|\psi\|^2$$

for any $\psi \in L^2(\boldsymbol{R}^{2\times N} \times Z^{a_{\max}})$, with $C > 0$ independent of ψ. Moreover, for any $\psi \in L^2(\boldsymbol{R}^{2\times N} \times Z^{a_{\max}})$ such that $(1 + |\Xi|_1)^{1/2}\psi \in L^2(\boldsymbol{R}^{2\times N} \times Z^{a_{\max}})$,

$$\int_1^\infty \left\| \mathbf{1}_{[M_1,\infty)}\left(\frac{|\Xi|_1}{t}\right) f(H)e^{-itH}\psi \right\|^2 \frac{dt}{t} < \infty$$

holds.

As for the proof, see [2].

Finally we prove the following minimal velocity estimate, which can be shown by virtue of the Mourre estimate in Corollary 2.2.

Theorem 3.5. *Let λ, δ, c and f be also as in Corollary 2.2. Then for any real-valued $h \in C_0^\infty(\boldsymbol{R})$, there exists $\varepsilon_0 > 0$ such that*

$$\int_1^\infty \left\| \mathbf{1}_{[0,\varepsilon_0]}\left(\frac{|\Xi|_1}{t}\right) f(H)h(N_0)e^{-itH}\psi \right\|^2 \frac{dt}{t} \leq C\|\psi\|^2$$

for any $\psi \in L^2(\boldsymbol{R}^{2\times N} \times Z^{a_{\max}})$, with $C > 0$ independent of ψ.

As for the proof, see [2].

4 Proof of Theorem 1.1

Throughout this section, we assume the conditions $(V.1)$, $(V.2)$, $(V.3)$ and (SR). First we prove the existence of the Deift-Simon wave operators

$$\check{W}_a^+ = \operatorname{s-lim}_{t\to\infty} e^{itH_a}\tilde{q}_a\left(\frac{\Xi}{t}\right)e^{-itH}, \quad a \in \mathcal{A}. \tag{4.1}$$

We note that N_0 commutes with H. By a density argument, for $\psi \in L^2(\boldsymbol{R}^{2\times N} \times Z^{a_{\max}})$ such that

$$\psi = f(H)\psi, \quad \psi = h(N_0)\psi$$

with $f,\ h \in C_0^\infty(\boldsymbol{R})$, we have only to prove the existence of

$$\check{W}_a^+\psi = \lim_{t\to\infty} e^{itH_a}\tilde{q}_a\left(\frac{\Xi}{t}\right)e^{-itH}\psi, \quad a \in \mathcal{A}. \tag{4.2}$$

In order to carry it out, by taking $f_1,\ h_1 \in C_0^\infty(\boldsymbol{R})$ such that $f_1 f = f$ and $h_1 h = h$, we have only to show the existence of

$$\lim_{t\to\infty} e^{itH_a}h_1(N_0)f_1(H_a)\tilde{q}_a\left(\frac{\Xi}{t}\right)e^{-itH}\psi, \quad a \in \mathcal{A}. \tag{4.3}$$

Here we note that

$$h_1(N_0)f_1(H_a)\tilde{q}_a\left(\frac{\Xi}{t}\right) - \tilde{q}_a\left(\frac{\Xi}{t}\right)f_1(H)h_1(N_0) = O(t^{\max\{-1,-\mu_{S1}\}}) = O(t^{-1}).$$

As is well-known, Proposition 3.4 implies

$$\operatorname{s-lim}_{t\to\infty}\left\{1 - J^2\left(\frac{\Xi}{t}\right)\right\}e^{-itH}f(H) = 0, \tag{4.4}$$

where $J \in C_0^\infty(\mathcal{X})$ be a cut-off function such that $J = 1$ on $\{\Xi \in \mathcal{X} \mid |\Xi|_1 \leq \theta\}$ and $J \geq 0$ with sufficiently large $\theta > 0$ (see e.g. [1]). By virtue of (4.4), we have only to show the existence of

$$\lim_{t\to\infty} e^{itH_a}h_1(N_0)f_1(H_a)J\left(\frac{\Xi}{t}\right)\tilde{q}_a\left(\frac{\Xi}{t}\right)J\left(\frac{\Xi}{t}\right)e^{-itH}\psi, \quad a \in \mathcal{A}. \tag{4.5}$$

The existence of (4.5) is proved by virtue of Theorem 3.3 and Proposition 3.4 (see [2] for the detail). Therefore we get the existence of the Deift-Simon wave operators $\check{W}_a^+$, $a \in \mathcal{A}$.

Using the same argument as the one to show the existence of the Deift-Simon wave operators $\check{W}_a^+$, $a \in \mathcal{A}$, one can prove the existence of the usual wave operators W_a^+, $a \in \mathcal{A}(a_{\max})$, which are defined by (1.17). For the detail, see [2]. We note that one can prove the closedness of the ranges of W_a^+, $a \in \mathcal{A}(a_{\max})$, their mutual orthogonality and

$$\sum_{a\in\mathcal{A}(a_{\max})} \oplus\, Ran\, W_a^\pm \subset L_c^2(H)$$

in the same way as in the case for many body Schrödinger operators without external electromagnetic fields.

Finally we prove the asymptotic completeness. We first claim that letting $f \in C_0^\infty(\boldsymbol{R})$ as in Corollary 2.2, we have for any real-valued $h \in C_0^\infty(\boldsymbol{R})$

$$\check{W}_{a_{\max}}^+ f(H)h(N_0) = 0 \tag{4.6}$$

with sufficiently small $r > 0$ in the definition of $\{\tilde{q}_a(\Xi) \mid a \in \mathcal{A}\}$. In fact, by virtue of Theorem 3.5, we have only to take $r > 0$ so small that $r < {\varepsilon_0}^2$, where $\varepsilon_0 > 0$ is as in Theorem 3.5.

Now we prove the asymptotic completeness by induction with respect to $N \geq 2$. First we note that in the case when $N = 2$, the asymptotic completeness was proved in [1]. Assume that the asymptotic completeness holds for M-body systems in which there exists only one charged particle with $2 \leq M < N$. By a density argument, we have only to consider $\psi \in L_c^2(H)$ such that

$$\psi = h(N_0)\psi, \quad \psi = f(H)\psi$$

with $h \in C_0^\infty(\boldsymbol{R})$ and $f \in C_0^\infty(\boldsymbol{R})$ as in Corollary 2.2. Here we also notice that $\Theta \cup \sigma_{\mathrm{pp}}(H)$ is a closed countable set (see Theorem 2.1). If we take $r > 0$ so small that $r < {\varepsilon_0}^2$, we see that

$$\begin{aligned} e^{-itH}\psi &= \sum_{a \in \mathcal{A}} \tilde{q}_a\left(\frac{\Xi}{t}\right) e^{-itH}\psi = \sum_{a \in \mathcal{A}(a_{\max})} e^{-itH_a}\check{W}_a^+\psi + o(1) \\ &= \sum_{a \in \mathcal{A}(a_{\max})} e^{-itH_a}P^a\check{W}_a^+\psi + \sum_{a \in \mathcal{A}(a_{\max})} e^{-itH_a}(Id - P^a)\check{W}_a^+\psi + o(1) \end{aligned} \tag{4.7}$$

as $t \to \infty$. Here we used Proposition 3.1, the existence of the Deift-Simon wave operators $\check{W}_a^+$, and (4.6). For any $\varepsilon > 0$, there exist a finite number of $\tilde{\psi}_j^a \in L^2(X^{a,\mathrm{n}})$, $\hat{\psi}_j^a \in L^2(\boldsymbol{R}^{2 \times \#(C_{\#(a)})} \times Z^{C_{\#(a)}})$, $\psi_{a,j} \in L^2(Y_{a,\mathrm{n}}) \otimes L^2(Z_a)$ such that

$$\left\| \check{W}_a^+\psi - \sum_{j:\text{finite}} \tilde{\psi}_j^a \otimes \hat{\psi}_j^a \otimes \psi_{a,j} \right\| < \varepsilon. \tag{4.8}$$

Now one can apply the asymptotic completeness for K^a and $H^{C_{\#(a)}}$, where we recall that K^a is an $(N - \#(C_{\#(a)}))$-body Schrödinger operator without external electromagnetic fields in the center of mass frame, and $H^{C_{\#(a)}}$ is the $\#(C_{\#(a)})$-body Hamiltonian under consideration. We also note that the asymptotic completeness for K^a under the condition (SR) was already obtained by several authors (see e.g. [22], [14] and [26]).

For $a = \{C_1, \ldots, C_{\#(a)}\} \in \mathcal{A}(a_{\max})$ with $\{N\} \subset C_{\#(a)}$, we put $a^{\mathrm{n}} = \{C_1, \ldots, C_{\#(a)-1}\}$ and $a^{\mathrm{c}} = \{C_{\#(a)}\}$. Let $\mathcal{A}_a^{\mathrm{n}}$ be the set of all cluster decompositions b^{n} of $\cup_{j=1}^{\#(a)-1} C_j$ such that $b^{\mathrm{n}} \subset a^{\mathrm{n}}$, and $\mathcal{A}_a^{\mathrm{c}}$ be the set of all cluster decompositions b^{c} of $C_{\#(a)}$ such that $b^{\mathrm{c}} \subset a^{\mathrm{c}}$.

Put $\mathcal{A}_a^{\rm n}(a^{\rm n}) = \mathcal{A}_a^{\rm n} \setminus \{a^{\rm n}\}$ and $\mathcal{A}_a^{\rm c}(a^{\rm c}) = \mathcal{A}_a^{\rm c} \setminus \{a^{\rm c}\}$. Taking account of that the asymptotic completeness for $H^{C_{\#(a)}}$, $a \in \mathcal{A}(a_{\max})$, holds by the assumption of induction, we have

$$Ran\,(Id - \tilde{P}^a) = \sum_{b^{\rm n} \in \mathcal{A}_a^{\rm n}(a^{\rm n})} \oplus\, Ran\, \tilde{W}^+(K^a, K^a_{b^{\rm n}}) \tag{4.9}$$

with

$$\tilde{W}^+(K^a, K^a_{b^{\rm n}}) = \operatorname*{s-lim}_{t\to\infty} e^{itK^a} e^{-itK^a_{b^{\rm n}}} \tilde{P}^a_{b^{\rm n}}$$

on $L^2(X^{a,{\rm n}})$, where $K^a_{b^{\rm n}} = K^a - \tilde{I}^{a^{\rm n}}_{b^{\rm n}}$ with

$$\tilde{I}^{a^{\rm n}}_{b^{\rm n}} = \sum_{\substack{(l_1,l_2)\subset a^{\rm n} \\ (l_1,l_2)\not\subset b^{\rm n}}} V_{l_1 l_2}(x_{l_1} - x_{l_2}),$$

$\tilde{P}^a_{b^{\rm n}} = \tilde{P}^a_{b^{\rm n}} \otimes Id$ is the eigenprojection for the subsystem Hamiltonian associated with $K^a_{b^{\rm n}}$, as well as

$$Ran\,(Id - \hat{P}^a) = \sum_{b^{\rm c} \in \mathcal{A}_a^{\rm c}(a^{\rm c})} \oplus\, Ran\, \hat{W}^+(H^{C_{\#(a)}}, H^{C_{\#(a)}}_{b^{\rm c}}) \tag{4.10}$$

with

$$\hat{W}^+(H^{C_{\#(a)}}, H^{C_{\#(a)}}_{b^{\rm c}}) = \operatorname*{s-lim}_{t\to\infty} e^{itH^{C_{\#(a)}}} e^{-itH^{C_{\#(a)}}_{b^{\rm c}}} \hat{P}^a_{b^{\rm c}}$$

on $L^2(\boldsymbol{R}^{2\times\#(C_{\#(a)})} \times Z^{C_{\#(a)}})$, where $H^{C_{\#(a)}}_{b^{\rm c}} = H^{C_{\#(a)}} - \hat{I}^{a^{\rm c}}_{b^{\rm c}}$ with

$$\hat{I}^{a^{\rm c}}_{b^{\rm c}} = \sum_{\substack{(l_1,l_2)\subset a^{\rm c} \\ (l_1,l_2)\not\subset b^{\rm c}}} V_{l_1 l_2}(x_{l_1} - x_{l_2}),$$

$\hat{P}^a_{b^{\rm c}}$ is the eigenprojection for $H^{C_{\#(a)}}_{b^{\rm c}}$, which is defined in the same way as P^a associated with H_a. Thus there exist $\tilde{\varphi}_{b^{\rm n},j} \in L^2(X^{a,{\rm n}})$, $b^{\rm n} \in \mathcal{A}_a^{\rm n}(a^{\rm n})$, such that

$$(Id - \tilde{P}^a)\tilde{\psi}^a_j = \sum_{b^{\rm n} \in \mathcal{A}_a^{\rm n}(a^{\rm n})} \tilde{W}^+(K^a, K^a_{b^{\rm n}})\tilde{\varphi}_{b^{\rm n},j} \tag{4.11}$$

by (4.9), and there exist $\hat{\varphi}_{b^{\rm c},j} \in L^2(\boldsymbol{R}^{2\times\#(C_{\#(a)})} \times Z^{C_{\#(a)}})$, $b^{\rm c} \in \mathcal{A}_a^{\rm c}(a^{\rm c})$, such that

$$(Id - \hat{P}^a)\hat{\psi}^a_j = \sum_{b^{\rm c} \in \mathcal{A}_a^{\rm c}(a^{\rm c})} \hat{W}^+(H^{C_{\#(a)}}, H^{C_{\#(a)}}_{b^{\rm c}})\hat{\varphi}_{b^{\rm c},j} \tag{4.12}$$

by (4.10). Thus, taking account of

$$Id \otimes Id - \tilde{P}^a \otimes \hat{P}^a = (Id - \tilde{P}^a) \otimes (Id - \hat{P}^a) + (Id - \tilde{P}^a) \otimes \hat{P}^a + \tilde{P}^a \otimes (Id - \hat{P}^a),$$

we have as $t \to \infty$

$$
\begin{aligned}
& e^{-itH}\psi \\
&= \sum_{a\in\mathcal{A}(a_{\max})} e^{-itH_a}P^a\check{W}_a^+\psi + o(1) + O(\varepsilon) \\
&\quad + \sum_{\substack{a\in\mathcal{A}(a_{\max})\\ j:\text{finite}}} \Bigg\{ \sum_{\substack{b^{\mathrm{n}}\in\mathcal{A}_a^{\mathrm{n}}(a^{\mathrm{n}})\\ b^{\mathrm{c}}\in\mathcal{A}_a^{\mathrm{c}}(a^{\mathrm{c}})}} e^{-itH_a}(\tilde{W}^+(K^a, K^a_{b^{\mathrm{n}}})\tilde{\varphi}_{b^{\mathrm{n}},j} \otimes \hat{W}^+(H^{C_{\#(a)}}, H_{b^{\mathrm{c}}}^{C_{\#(a)}})\hat{\varphi}_{b^{\mathrm{c}},j} \otimes \psi_{a,j}) \\
&\qquad\qquad + \sum_{b^{\mathrm{n}}\in\mathcal{A}_a^{\mathrm{n}}(a^{\mathrm{n}})} e^{-itH_a}(\tilde{W}^+(K^a, K^a_{b^{\mathrm{n}}})\tilde{\varphi}_{b^{\mathrm{n}},j} \otimes \hat{P}^a\hat{\psi}_j^a \otimes \psi_{a,j}) \\
&\qquad\qquad + \sum_{b^{\mathrm{c}}\in\mathcal{A}_a^{\mathrm{c}}(a^{\mathrm{c}})} e^{-itH_a}(\tilde{P}^a\tilde{\psi}_j^a \otimes \hat{W}^+(H^{C_{\#(a)}}, H_{b^{\mathrm{c}}}^{C_{\#(a)}})\hat{\varphi}_{b^{\mathrm{c}},j} \otimes \psi_{a,j}) \Bigg\}.
\end{aligned}
\tag{4.13}
$$

For $b^{\mathrm{n}} = \{B_1^{\mathrm{n}}, \dots, B^{\mathrm{n}}_{\#(b^{\mathrm{n}})}\} \in \mathcal{A}_a^{\mathrm{n}}$ and $b^{\mathrm{c}} = \{B_1^{\mathrm{c}}, \dots, B^{\mathrm{c}}_{\#(b^{\mathrm{c}})}\} \in \mathcal{A}_a^{\mathrm{c}}$, we write

$$
b^{\mathrm{n}} + b^{\mathrm{c}} = \{B_1^{\mathrm{n}}, \dots, B^{\mathrm{n}}_{\#(b^{\mathrm{n}})}, B_1^{\mathrm{c}}, \dots, B^{\mathrm{c}}_{\#(b^{\mathrm{c}})}\} \in \mathcal{A}_a = \{b \in \mathcal{A} \mid b \subset a\}.
$$

We note that, for $b^{\mathrm{n}} \in \mathcal{A}_a^{\mathrm{n}}$ and $b^{\mathrm{c}} \in \mathcal{A}_a^{\mathrm{c}}$, we see that $b^{\mathrm{n}} + b^{\mathrm{c}}$, $b^{\mathrm{n}} + a^{\mathrm{c}}$, $a^{\mathrm{n}} + b^{\mathrm{c}} \in \mathcal{A}(a) = \{b_1 \in \mathcal{A} \mid b_1 \subsetneq a\} = \mathcal{A}_a \setminus \{a\}$. Taking account of the definition of $\tilde{W}^+(K^a, K^a_{b^{\mathrm{n}}})$ and $\hat{W}^+(H^{C_{\#(a)}}, H_{b^{\mathrm{c}}}^{C_{\#(a)}})$, and rearranging some terms in (4.13) with respect to $b \in \mathcal{A}(a)$, we have as $t \to \infty$

$$
e^{-itH}\psi = \sum_{a\in\mathcal{A}(a_{\max})} e^{-itH_a}P^a\check{W}_a^+\psi + \sum_{\substack{a\in\mathcal{A}(a_{\max})\\ j:\text{finite}}} \sum_{b\in\mathcal{A}(a)} e^{-itH_b}P^b(\psi_j^b \otimes \psi_{a,j}) + o(1) + O(\varepsilon)
\tag{4.14}
$$

with some $\psi_j^b \in L^2(X^{a,\mathrm{n}}) \otimes L^2(\boldsymbol{R}^{2\times\#(C_{\#(a)})} \times Z^{C_{\#(a)}})$. Multiplying both sides of (4.14) by e^{itH} and taking $t \to \infty$, we have

$$
\psi = \sum_{a\in\mathcal{A}(a_{\max})} W_a^+\check{W}_a^+\psi + \sum_{\substack{a\in\mathcal{A}(a_{\max})\\ j:\text{finite}}} \sum_{b\in\mathcal{A}(a)} W_b^+(\psi_j^b \otimes \psi_{a,j}) + O(\varepsilon).
\tag{4.15}
$$

Since one can take $\varepsilon > 0$ arbitrary, this implies

$$
\psi \in \sum_{a\in\mathcal{A}(a_{\max})} \oplus Ran\, W_a^+,
$$

by virtue of the closedness of the ranges of W_a^+, $a \in \mathcal{A}(a_{\max})$. The proof is completed. $\square$

参考文献

[1] T.Adachi, Scattering theory for a two-body quantum system in a constant magnetic field, J. Math. Sci., The Univ. of Tokyo **8** (2001), 243–274.

[2] T.Adachi, On spectral and scattering theory for N-body Schrödinger operators in a constant magnetic field, to appear in Rev. Math. Phys..

[3] T.Adachi, On the Mourre estimates for N-body Schrödinger operators in a constant magnetic field, preprint.

[4] J.Avron, I.W.Herbst and B.Simon, Schrödinger operators with magnetic fields. I. General interactions, Duke Math. J. **45** (1978), 847–883.

[5] J.Avron, I.W.Herbst and B.Simon, Separation of center of mass in homogeneous magnetic fields, Ann. Phys. **114** (1978), 431–451.

[6] H.Cycon, R.G.Froese, W.Kirsch and B.Simon, Schrödinger Operators with Application to Quantum Mechanics and Global Geometry, Texts and Monographs in Physics, Springer-Verlag 1987.

[7] J.Dereziński, Asymptotic completeness of long-range N-body quantum systems, Ann. of Math. **138** (1993), 427–476.

[8] J.Dereziński and C.Gérard, Scattering Theory of Classical and Quantum N-Particle Systems, Springer-Verlag 1997.

[9] R.Froese and I.W.Herbst, A new proof of the Mourre estimate, Duke Math. J. **49** (1982), 1075–1085.

[10] C.Gérard and I.Łaba, Scattering theory for N-particle systems in constant magnetic fields, Duke Math. J. **76** (1994), 433–465.

[11] C.Gérard and I.Łaba, Scattering theory for N-particle systems in constant magnetic fields, II. Long-range interactions, Commun. P. D. E. **20** (1995), 1791–1830.

[12] C.Gérard and I.Łaba, Scattering theory for 3-particle systems in constant magnetic fields: Dispersive case, Ann. Inst. Fourier, Grenoble **46** (1996), 801–876.

[13] C.Gérard and I.Łaba, Multiparticle Quantum Scattering in Constant Magnetic Fields, Mathematical Surveys and Monographs, AMS 2002.

[14] G.M.Graf, Asymptotic completeness for N-body short-range quantum systems: a new proof, Commun. Math. Phys. **132** (1990), 73–101.

[15] B.Helffer and J.Sjöstrand, Equation de Schrödinger avec champ magnétique et équation de Harper, Lecture Notes in Physics 345, Springer-Verlag 1989, 118–197.

[16] A.Jensen and S.Nakamura, The 2D Schrödinger equation for a neutral pair in a constant magnetic field, Ann. Inst. Henri Poincaré - Phys. Théor. **67** (1997), 387–410.

[17] I.Łaba, Scattering for hydrogen-like systems in a constant magnetic field, Commun. P. D. E. **20** (1995), 741–762.

[18] I.Łaba, Multiparticle quantum systems in constant magnetic fields, Multiparticle quantum scattering with applications to nuclear, atomic and molecular physics (Minneapolis, MN, 1995), IMA Vol. Math. Appl., 89, Springer-Verlag 1997, 147–215.

[19] E.Mourre, Absence of singular continuous spectrum for certain self-adjoint operators, Commun. Math. Phys. **78** (1981), 391–408.

[20] P.Perry, I.M.Sigal and B.Simon, Spectral analysis of N-body Schrödinger operators, Ann. of Math. **114** (1981), 517–567.

[21] M.Reed and B.Simon, Methods of Modern Mathematical Physics I–IV, Academic Press.

[22] I.M.Sigal and A.Soffer, The N-particle scattering problem: asymptotic completeness for short-range systems, Ann. of Math. **125** (1987), 35–108.

[23] I.M.Sigal and A.Soffer, Long-range many body scattering: Asymptotic clustering for Coulomb type potentials, Invent. Math. **99** (1990), 115–143.

[24] E.Skibsted, On the asymptotic completeness for particles in constant electromagnetic fields, Partial differential equations and mathematical physics (Copenhagen, 1995; Lund, 1995), Progr. Nonlinear Differential Equations Appl., 21, Birkhäuser 1996, 286–320.

[25] E.Skibsted, Asymptotic completeness for particles in combined constant electric and magnetic fields, II, Duke Math. J. **89** (1997), 307–350.

[26] D.Yafaev, Radiation conditions and scattering theory for N-particle Hamiltonians, Commun. Math. Phys. **154** (1993), 523–554.